SPLIT-RADIX FAST HARTLEY TRANSFORM

Indexing terms: Signal processing, Fast Fourier transforms, Transforms

The split radix is used to develop a fast Hartley transform algorithm; it is performed 'in-place', and requires the lowest number of arithmetic operations compared with other related algorithms.

Introduction: Bracewell1-2 recently proposed a fast Hartley transform (FHT). This transform is closely related to the fast Fourier transform (FFT). It has two advantages over the FFT, however: (i) the forward and the inverse transforms are the same; (ii) the Hartley transformed outputs are real-valued rather than complex data, as with the FFT; also the Fourier spectrum can be calculated via the Hartley transform.

The split radix has been proposed for FFT computations by Duhamel and Hollmann.3 This algorithm has the advantage of being performed 'in-place' in an FFT-like structure, and requires the lowest number of multiplications and additions for length $N = 2^q$.

In this letter we use split-radix decomposition to develop a fast Hartley transform; it is shown to be faster (in terms of multiplication and addition counts) than other current algorithms.

Split-radix algorithm: The split-radix algorithm4 is based on both even-term radix-2 decompositions and odd-term radix-4 decompositions simultaneously. If the discrete Hartley transform (DHT)

$$X_k = \sum_{n=0}^{N-1} x_n \left( \cos \frac{2\pi}{N} nk + \sin \frac{2\pi}{N} nk \right)$$

is to be computed, it is decomposed as

$$X_{2k} = \sum_{n=0}^{(N/2)-1} \left( x_n + x_{n+(N/2)} \right) \cos \frac{2\pi}{N/2} nk + \sin \frac{2\pi}{N/2} nk$$

$$X_{4k+1} = \sum_{n=0}^{(N/4)-1} \left( A_n + A_{(N/4)-n} \right) \cos \frac{2\pi}{N/4} 3n$$

$$- \left( B_n - B_{(N/4)-n} \right) \sin \frac{2\pi}{N} n$$

$$X_{4k+3} = \sum_{n=0}^{(N/4)-1} \left( A_n - A_{(N/4)-n} \right) \cos \frac{2\pi}{N/4} 3n$$

$$+ \left( B_n + B_{(N/4)-n} \right) \sin \frac{2\pi}{N} n$$

where

$$A_u = x_u - x_{u+(N/2)}$$

$$B_u = x_u + x_{u+(N/2)}$$

and

$$A_{(N/4)} = B_0$$

The above decimation-in-frequency decomposition reduces an $N$-point DHT into one $(N/2)$-point DHT and two $(N/4)$-point DHTs at the cost of $(N - 4)$ real multiplications and $(2N - 4)$ real additions, i.e.

$$N$$-point DHT $\rightarrow N \rightarrow 2 \rightarrow \frac{N}{4}$$-point DHTs

cost: $(N - 4)$ real 'multiplys' and $(2N - 4)$ real 'adds'

Fig. 1 shows the flow graph of the decomposition of a 16-point DHT computation into one 8-point DHT and two 4-point DHTs. The N-point DHT is then obtained by successive use of such decompositions up to the last stage. Fig. 2 shows a complete 8-point FHT algorithm. All decimation-in-frequency FHT algorithms provide the output points in bit-reversal order.

Computational complexity: The total numbers of operations of the split-radix FHT are

$$A = (2N - 4) \cdot \lceil \log_4 N \rceil$$

$$M = (N - 4) \cdot \lceil \log_4 N \rceil$$

(split-radix FHT)

where $\lceil u \rceil$ denotes the smallest integer greater than or equal to $u$. Compare this with other current algorithms:

$$A = 2N \cdot (\log_2 N - 1) + 2 \left( \text{Bracewell}^2 \right)$$

$$M = N \cdot (\log_2 N - 2) + 2 \left( \text{decimation-in-time FHT} \right)$$

and

$$A = \frac{3}{2} N \cdot (\log_2 N - 1) + 2 \left( \text{Meckelburg and Lipka}^4 \text{decimation-in-frequency FHT} \right)$$

In Tables 1 and 2 we list the number of multiplications and additions required to compute an $N$-point DHT. The split-radix algorithm has the lowest number of multiplications and additions compared with other related algorithms.

The split-radix algorithm needs less stages $\lceil \log_4 N \rceil$ than $\log_2 N$ in a conventional radix-2 DHT. This results in a much smaller number of operations.
lower required number of arithmetic operations. Since the DHT is a symmetrical transform, additional savings can be gained (see Fig. 3):

(a) When
\[
\theta = \frac{2\pi nk}{N} = \frac{1}{4} \pi \text{ or } \frac{5}{4} \pi \quad ((2N - 4)/N^2 \text{ to occur})
\]
\[
\cos \theta + \sin \theta = \sqrt{2}/2 \text{ or } -\sqrt{2}
\]
(save one multiplication)

(b) When
\[
\theta = \frac{2\pi nk}{N} = \frac{3}{4} \pi \text{ or } \frac{7}{4} \pi \quad ((2N - 4)/N^2 \text{ to occur})
\]
\[
\cos \theta + \sin \theta = 0
\]
(save 2 multiplications and 1 addition)

The practical problem is how to detect these additional savings without introducing too much complexity. This requires further study and research.

Table 1 NUMBER OF REAL MULTIPLICATIONS TO COMPUTE AN N-POINT DHT

<table>
<thead>
<tr>
<th>N</th>
<th>Bracewell</th>
<th>Meckelburg-Lipka</th>
<th>Split-radix</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>34</td>
<td>20</td>
<td>24</td>
</tr>
<tr>
<td>32</td>
<td>98</td>
<td>68</td>
<td>84</td>
</tr>
<tr>
<td>64</td>
<td>258</td>
<td>196</td>
<td>180</td>
</tr>
<tr>
<td>128</td>
<td>642</td>
<td>516</td>
<td>496</td>
</tr>
<tr>
<td>256</td>
<td>1538</td>
<td>1284</td>
<td>1008</td>
</tr>
<tr>
<td>512</td>
<td>3586</td>
<td>3076</td>
<td>2540</td>
</tr>
<tr>
<td>1024</td>
<td>8194</td>
<td>7172</td>
<td>5100</td>
</tr>
</tbody>
</table>

Table 2 NUMBER OF REAL ADDITIONS TO COMPUTE AN N-POINT DHT

<table>
<thead>
<tr>
<th>N</th>
<th>Bracewell</th>
<th>Meckelburg-Lipka</th>
<th>Split-radix</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>98</td>
<td>74</td>
<td>56</td>
</tr>
<tr>
<td>32</td>
<td>258</td>
<td>194</td>
<td>180</td>
</tr>
<tr>
<td>64</td>
<td>642</td>
<td>482</td>
<td>372</td>
</tr>
<tr>
<td>128</td>
<td>1538</td>
<td>1154</td>
<td>1008</td>
</tr>
<tr>
<td>256</td>
<td>3586</td>
<td>2690</td>
<td>2032</td>
</tr>
<tr>
<td>512</td>
<td>8194</td>
<td>6146</td>
<td>5100</td>
</tr>
<tr>
<td>1024</td>
<td>18434</td>
<td>13826</td>
<td>10220</td>
</tr>
</tbody>
</table>

Fig. 3 Additional DHT computational savings

Conclusion: The split-radix algorithm is presented for the fast computation of the discrete Hartley transform. This decomposition combines radix-2 flexibility and radix-4 regularity in this algorithm, it is performed ‘in-place’ and requires the lowest number of arithmetic operations compared with other related algorithms.

SOO-CHANG PEI
Department of Electrical Engineering
National Taiwan University
Taipei, Taiwan, Republic of China

JA-LING WU
Department of Electrical Engineering
Tatung Institute of Technology
Taipei, Taiwan, Republic of China

References

PROPERTIES OF MODES ON PERTURBED FIBRES

Indexing terms: Optical fibres, Optical waveguides

The terms in the Fourier series for the radius of a perturbed step-index optical fibre dictate which particular modes become truly LP and determine their optical axes; these axes need not coincide for different modes. Nonlinear optics experiments may be used to measure the perturbations.

The design and use of optical fibres requires an understanding of the properties of the modal fields. Snyder\(^1\) showed that when core and cladding refractive indices, \(n_o\) and \(n_c\), differ only slightly, a simple, accurate analysis follows in the \(\Delta \approx (n_o - n_c)/n_o \approx 0\) limit. Gloge\(^2\) exploited that analysis and introduced the linearly polarised LP modes, which are a consequence of a degeneracy when \(\Delta \rightarrow 0\). For finite \(\Delta\), the broken degeneracy precludes the use of LP modes for accurate calculations if polarisation effect are important or unless fibre lengths are short, for example, in photoreceptor optics.\(^3\) An LP form for the transverse field \(E\) of a mode with propagation constant \(\beta\) follows from the vector equation

\[
\{V^2 + k^2 n^2 - \beta^2\}E = -V(E \cdot V \ln n^2)
\]

if the right-hand side is negligible.\(^4\)\(^-5\) \(k\) is the wavenumber and \(V\) is the transverse gradient operator. Then \(E = \psi(x, y)e\), where \(e\) is a unit polarisation vector and \(\psi\) obeys the scalar wave equation.

Two authors\(^6\)\(^-7\) incorporated the effects of the neglected term on the right-hand side of eqn. 1 to correct the LP-plus-scalar-field scheme and to show when it is appropriate. We apply the theory to a step-index fibre with an arbitrary perturbed core/cladding boundary and indicate how the perturbations might be measured in the few-mode case.

Problem: A step fibre with radius \(\rho_0\) and waveguide parameter \(V = \rho_0 k n_o \sqrt{(2\Delta)}\) is perturbed to have radius \(\rho\):

\[
\rho = \rho_0 \left\{1 + \sum_{j=1}^{n} a_j \cos \mathbf{\theta} \right\}
\]

Polar angles \(\theta_j\) and amplitudes \(a_j\) are arbitrary, but \(a_j\) is assumed small, \(a_j^2\) is negligible and perturbation methods\(^6\)\(^-7\) apply. Axes \(x_j', y_j'\) (Fig. 1) have unit vectors

\[
x_j' = \hat{x} \sin \theta_j + \hat{z} \cos \theta_j
\]
\[
y_j' = \hat{y} \cos \theta_j - \hat{x} \sin \theta_j
\]

Fundamental, \(l = 0\) modes: To first order, \(\beta\) is unchanged for the scalar field, but corrections to the field \(\psi\), for example by the Green function method,\(^9\) involves terms proportional to \(a_1 \cos (\mathbf{\theta})\). The optical axes are found by using \(\psi\) in the standard procedure (Appendix A of Reference 6 or Section 32-5 of Reference 8) for determining \(\beta\) corrections due to the right-hand side of eqn. 1. By symmetry, only the term in \(\psi\) proportional to \(a_2\) has any effect; the fundamental modes have the form \(\psi(x, y)\{x_j'\}^2\) or \(y_j'\). The birefringence depends only on the \(a_j\) or 'elliptical' term in the perturbation, and is therefore already known; see, for example, Section 18-10 of Reference 8.